GENERAL ARTICLE

Ascending in Space Dimensions: Digital Crafting of M.C. Escher's Graphic Art

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ABSTRACT

M.C. Escher's artwork has inspired and arguably even informed computer science, as well as geometric and shape modeling. Even today, much of his work poses challenges to conventional digital shape modeling systems. The authors introduce several interesting problems presented by Escher's graphic artworks and describe their use of a novel approach, based on implicit surfaces and their extension (Function Representation), to produce 2D, 2.5D and 3D computer models. They also discuss several physical objects or sculptures based on these models, crafted using digital fabrication processes.

This work was inspired by graphic artworks and prints by the Dutch artist M.C. Escher. We were attracted by his imaginative flair and attracted to the phantasmagorical yet geometrically based worlds of his graphic artworks because they present many challenges to computer modeling.

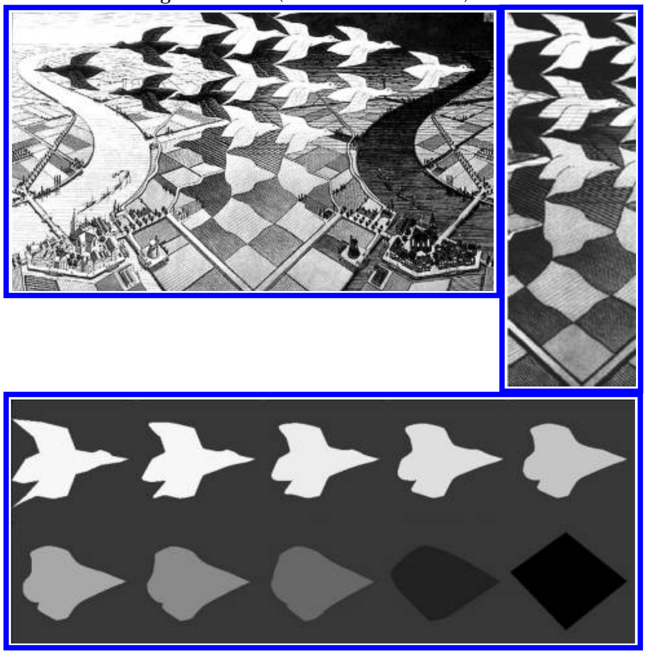
Escher's artworks are an ongoing source of inspiration for various computer-based projects. Kaplan provided a classification of Escher's approach to metamorphosis [1]. In the work of Savransky et al. [2] and Elber [3], drawings of so-called impossible objects are modeled in three dimensions and then fabricated as actual physical objects using rapid prototyping machines. The idea of these projects is that the computer models and physically produced objects resemble Escher's drawn objects when viewed from a certain direction. The "Escherization" project [4] is devoted to automatic discoveries of shape patterns for plane tiling. A similar problem is addressed in the Escher Sphere Construction Kit [5], where the authors develop interactive software for design and fabrication of tile sets to be assembled into a 3D ball.

We have identified the following problems, having no obvious solutions, in geometric modeling and computer graphics: the metamorphosis between two-dimensional (2D) polygons similar to shape transformations in *Day and Night* (1938), the construction of 2.5D shapes (bas-reliefs) from 2D projections as presented in Escher's *Reptiles* (1943) and the modeling of the complex spiral 3D shells depicted in Escher's *Rind* (1955) and other graphic artworks.

In our work, we try not only to reproduce the 2D graphic artworks, but also to bring to reality the interplay between spatial dimensions, so pivotal in much of Escher's work. Indeed, this interplay between a flat surface and a realistic emergent form is the theme of his lithograph *Hands* (1948). This artwork also reflects the idea of making flat images come alive and even the idea of self-replication by machines, an idea popular in the 3D printing community.

M.C. Escher's compositions are complex. His designs ascend from 2D drawings to 3D figures emerging from a plane, 3D tilings and impossible 3D objects. Escher also manually produced some 3D sculptures that are technically unlike our

Fig. 1. Shape metamorphosis: (top left) M.C. Escher's *Day and Night*, 39.1 × 67.7 cm, © 2011 The M.C. Escher Company-Holland. All rights reserved. <www.mcescher.com>; (right) the original transformation (a fragment of Escher's *Day and Night*); (bottom) a polygonal shape transformation using interpolation between defining functions. (© Alexander Pasko)



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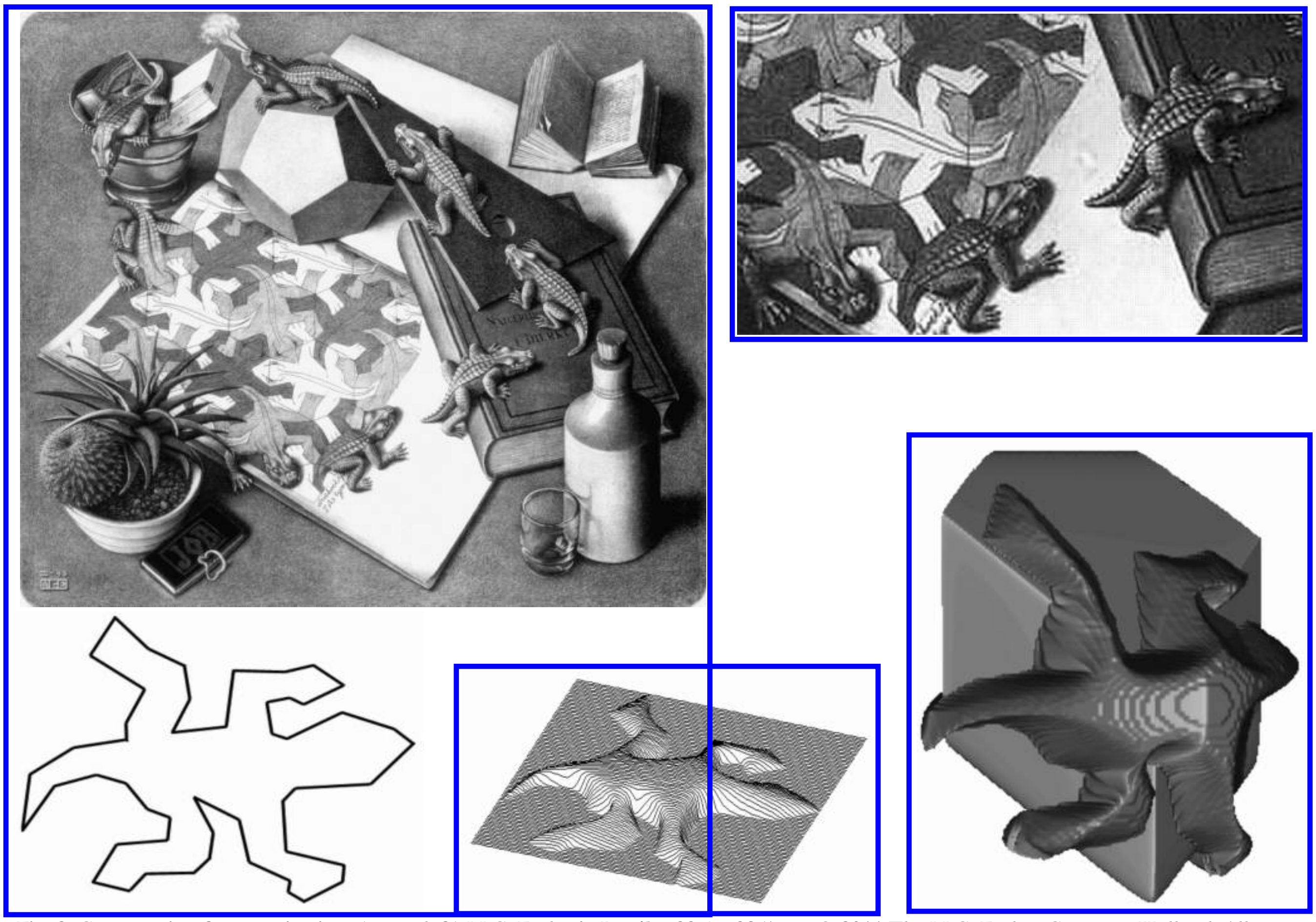


Fig. 2. Construction from projection: (a, top left) M.C. Escher's *Reptiles*, 33.4 × 38.5 cm, © 2011 The M.C. Escher Company-Holland. All rights reserved. <www.mcescher.com>; (b, top right) a fragment of Escher's *Reptiles* used for the model construction; (c, bottom left) an initial concave polygon as a reptile boundary; (d, bottom center) a height field inside the polygon generated using the polygon-to-function conversion procedure; (e, bottom right) a reptile relief extruded on the block surface using offsetting controlled by the height field data. (2c, d, e: © Alexander Pasko)

computer-facilitated works. Yet in some sense one could argue that the geometric modeling problems tackled in this paper also ascend in space dimensions from purely 2D drawings to 2.5D bas-reliefs, then to 3D spiral shell models and finally to physical fabrication of tangible objects.

To tackle the problems outlined above, we represented geometric shapes using the implicit surfaces model and its extension, called Function Representation (FRep). The shapes are represented with continuous functions of point sets, where the function changes its sign on either side of the object's boundary and is zero at boundary points. The FRep approach to geometric modeling was introduced by the HyperFun project and research team [6] as a generalization of implicit surfaces and constructive solid geometry and other existing approaches in geometric and solid modeling. The details of FRep and related publications can be found at a dedicated web site [7]. We implemented the proposed algorithms in the HyperFun language and rendered using the supporting software tools [8].

This paper is organized as follows: In the next section we present an approach to 2D shape metamorphosis; 3D shape construction from a projection is described in the section Construction from Projection, followed by a section on modeling complex shells. We also include a section on digital fabrication of introduced models. All the algorithms presented are illustrated by our experiments.

Shape Metamorphosis

A visually smooth transformation (or metamorphosis) between two arbitrary shapes is a non-trivial geometric modeling problem. A general solution can be found if both shapes are represented by real functions and the transformation is described as an interpolation between these functions. To find a solution to this problem, we first needed to represent a 2D polygon (convex or concave) using a real function of 2D point coordinates taking zero value at polygon edges. The algorithm should provide the exact polygonal boundary described as the zero set of a real-valued function. No points with function value of zero should exist inside or outside of the polygon.

Rvachev [9] proposed representing a concave polygon with a set-theoretic expression called a monotone formula, involving union and intersection operations on the half-planes passing through the polygon edges. Each of the half-planes appears exactly once, and no additional half-plane is used. The formula for the function that defines a polygon can be obtained from the set-theoretic expression, replacing each half-plane by its defining function while set-theoretic operations are replaced by the corresponding R-functions; see details in Pasko and Adzhiev [10].

We implemented the above approach [11] and applied it to several geometric modeling problems, including shape metamorphosis and construction from projections. Figure 1a shows Escher's print *Day and Night* and Fig. 1b shows its fragment, wherein a bird's outline is transformed into a quadrilateral. Our reproduction of this transformation is illustrated by Fig. 1c. Here, both polygons, the bird shape and the rectangle, are represented by corresponding real functions. The polygon representing the bird contour was converted to a real function using the monotone formula. The linear interpolation between two functions determines the in-between shapes. In the case that two functions have different behaviors (such as different decreasing speed), the linear interpolation gives poor results due to abrupt change from one shape to another. In that case, the function normalization or pure distance functions can be applied.

In addition to the six transformative steps Escher used, an arbitrary number of steps can be produced for the transformation. In this case we added four additional steps. It is worth noting the fair resemblance between several of the computer-generated intermediate shapes in Fig. 1b and the original shapes in *Day and Night* (Fig. 1a). Escher, in fact, may have been naturally or intentionally interpolating linearly between shapes.

Construction from Projection

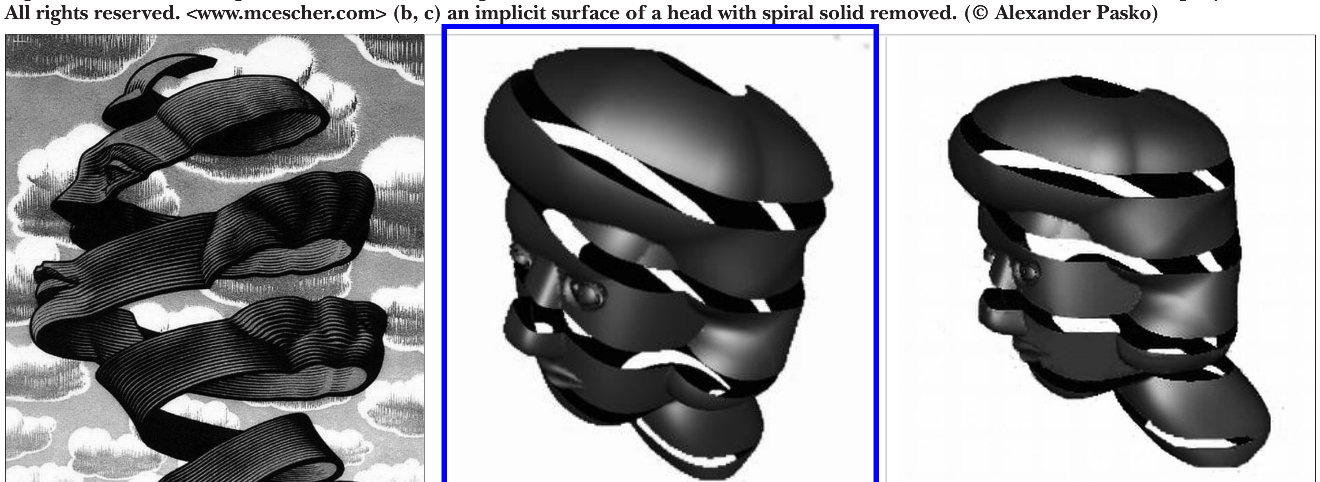
The problem of constructing 3D shapes from 2D projections is visually stated in the *Reptiles* lithograph by Escher (Fig. 2a) and especially in its fragment (Fig. 2b), where reptiles emerge into 3D space from their 2D images. To automate such a construction we employ the above- described functionally represented polygons as illustrated in Figs. 2c-e.

The first step is to extract a 2D polygon outlining the projection. In Fig. 2c a polygon with concave and convex features outlining the reptile's 2D boundary is shown. This polygon is then used to generate depth data or a height field inside the polygon (Fig. 2d). The height field can be considered a 2.5D model of the 2D polygon. The height field was generated by taking only the positive values of the defining function for the polygon (i.e. for the points inside the polygon) provided by the monotone formula mentioned above. The height values for points outside the polygon (corresponding to negative function values) are set to zero.

The generated height field represents a relief that can be extruded on a support object surface. The reptiles emerging from a notebook and climbing a book in Fig. 2a can be modeled in this way. In geometric modeling, an offsetting operation expands or shrinks the initial model by the given distance between the initial and final surfaces. We apply a non-uniform offset of the given support object in the direction of the object's normal vector at each surface point [12]. The offset distance is modulated by the placement and the value of the reptile's height field. Figure 2e shows a block with the reptile relief extruded from its surface using offsetting along the normal vector. Such a non-uniform offsetting can be considered one of the techniques of computer-aided relief carving [13], in which the offsetting along the normal controlled by height

Fig. 3. Cutting spiral threads from a spherical surface: (a) M.C. Escher's *Sphere Spirals*, 32 × 32 cm, © 2011 The M.C. Escher Company-Holland. All rights reserved. <www.mcescher.com> (© 2009 The M.C. Escher Company B.V., the Netherlands); (b) spiral solid; (c) sphere surface with spiral solid removed using set-theoretic subtraction operation. (3b, c: © Alexander Pasko)

Fig. 4. Creation of a complex helical shell: (a) a fragment of M.C. Escher's *Rind*, 34.5 × 23.5 cm, © 2011 The M.C. Escher Company-Holland. All rights reserved. <www.mcescher.com> (b, c) an implicit surface of a head with spiral solid removed. (© Alexander Pasko)



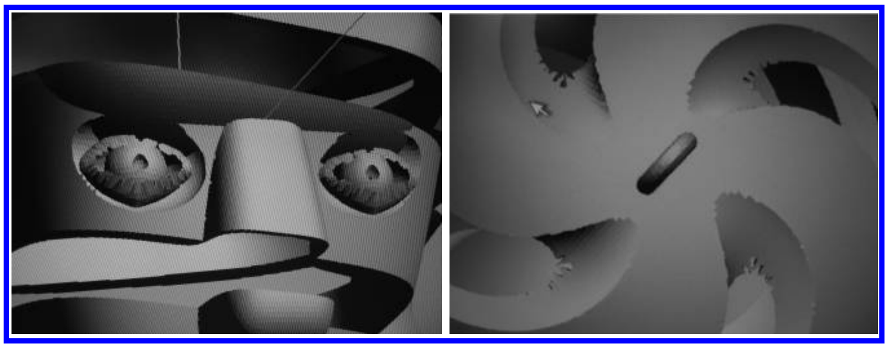


Fig. 5. (a,b) Some faults produced during STL export. (© Alexander Pasko)

fields can be applied to practically any initial FRep shape.

Modeling Complex Spiral Shells

This part of our work was inspired by several Escher graphic artworks, namely *Sphere Spirals* (1958), shown in Fig. 3a, *Bond of Union* (1956) and *Rind* (1955) (Fig. 4a), showing spiral threads cut from the surfaces of a sphere (in *Sphere Spirals*) and a human head (in *Rind*). These artworks raised the question for us of the ways in which one can define and visualize a geometric model of this type with current approaches based on surface patches.

A surface patch can conventionally be defined as a 2D manifold with boundary (2-manifold). Although modeling and visualization of the above-mentioned spherical spiral using conventional 2-manifolds can be an easy task, removing such a shape from any arbitrary shape, such as a human face, and creating an error-free polygon mesh can be a very difficult task for conventional modeling systems. An alternative is to use isosurfaces of functions of three variables (implicit surfaces). A surface patch can be represented as a set-theoretic difference between some initial implicit surface and the solid to be removed. Thus, for modeling Escher's *Sphere Spirals* a solid consisting of a union of four spirals can be introduced as shown in Fig. 3b. The operation of the set-theoretic difference between a spherical surface and this solid produces spiral spherical patches connected at two points at the top and bottom of the sphere (Fig. 3c). During polygonization for visualization, we applied a special algorithm for removing parts of implicit surfaces [14], which generates a triangle mesh that adapts to surfaces limited by objects' intersection curves.

We produced another example after Escher's *Rind* (1955) (see Fig. 4). The initial FRep solid (human head model) was constructed similarly to the Lego style, using simple building blocks (such as ellipsoids, free-form blobby shapes and tubular shapes following skeleton curves) and operations on them (union, intersection, difference). This constructive modeling is supported by the HyperFun language, which we used to create the 3D models shown.

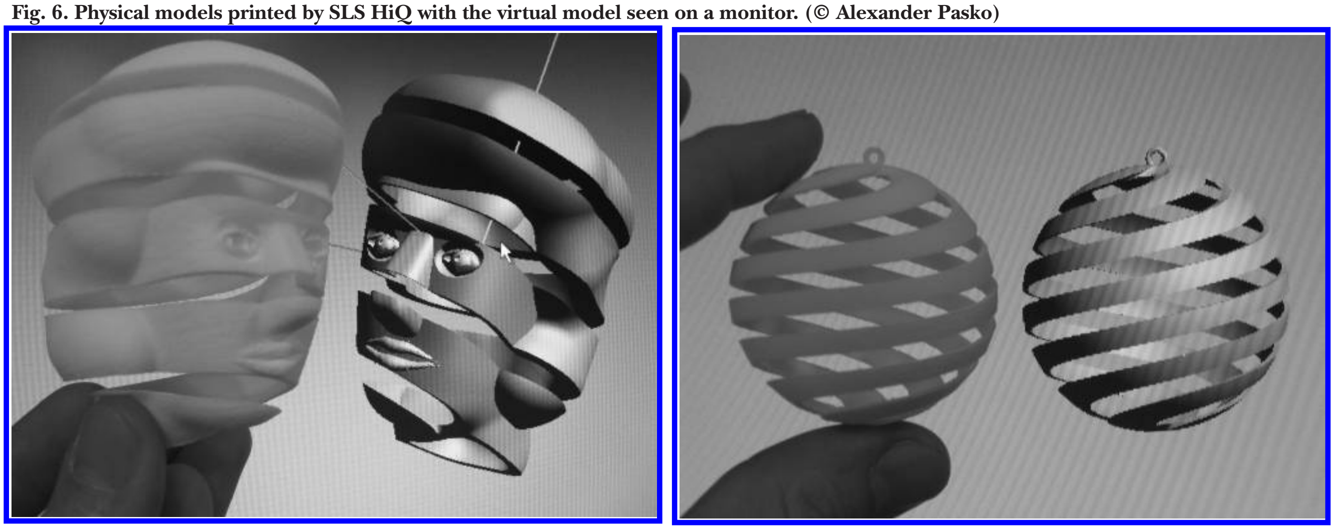
The parts of the implicit surface were removed from the head using a similar solid object as in the previous example, resulting in a ribbon-like helical shell. Causing the intersection of the head shape with the helical spirals would have produced a similar result. Note that the head's shell is rendered from two viewpoints (Figs 4b and 4c) to show the 3D nature of the obtained model. In contrast to this surface model, a thin shell of this object is needed for the fabrication purposes set out in the next section. We can produce a contracted version of the head with an offsetting operation and then remove this smaller head from the original model, which results in a thin shell.

Digital Fabrication

It is clear that Escher's graphic artworks, most of which deal with 3D space, can be made into interesting sculptures. After virtually modeling and visualizing Escher's work as "real 3D objects," we chose to digitally fabricate two of our models: *Sphere Spirals* and *Rind*. Digital fabrication (DF) is a technology that can reproduce a physical "touch-and-feel" model directly from a virtual computer model using a variety of processes and materials. This set of technology is also known as rapid prototyping, solid freeform fabrication, 3D printing, layered manufacturing and, more recently, as additive manufacturing (AM), as defined by the F.42 Committee of ASTM International.

Unlike conventional production processes, such as milling, AM enables freeform production of objects in almost any shape and form. The main idea is that the material is added layer by layer and is then bound to the object being built using various means, such as gluing, melting by heat, laser sintering and so on. AM

Fig. 6. Physical models printed by SLS HiQ with the virtual model seen on a monitor. (© Alexander Pasko)



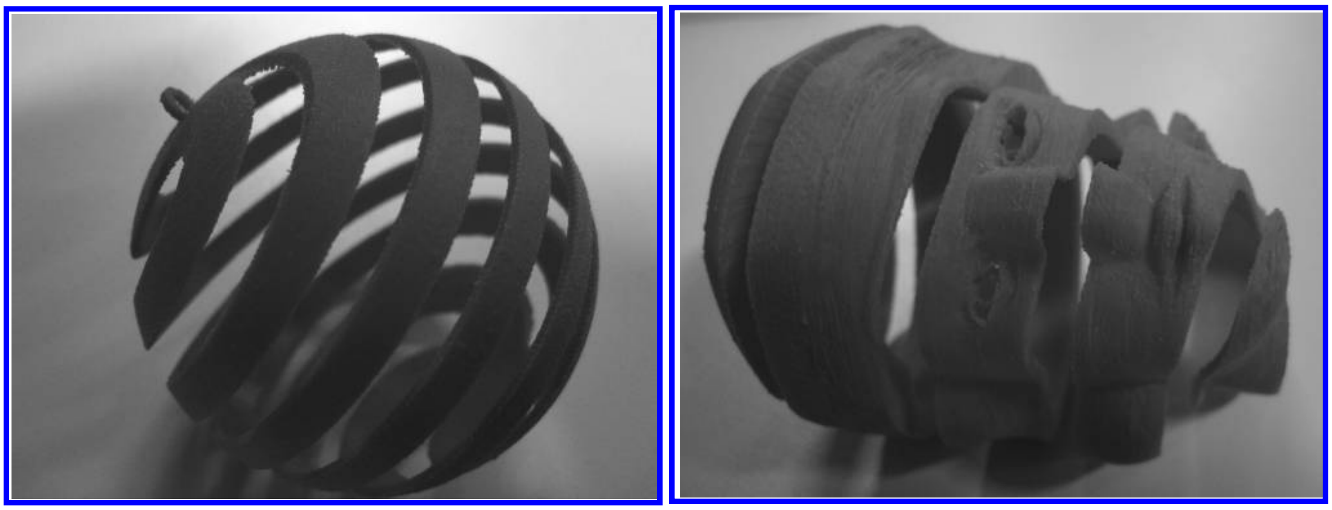


Fig. 7. Physical models printed by FDM Vantage I. (© Alexander Pasko)

is currently used in a myriad of different domains, such as surgical planning [15], architecture and product development, among many others. For artists, in particular, this can be a fantastic tool for bringing artistic creations to life in a way that may not have been possible otherwise.

The starting point of any current AM process is importing a 3D model represented in a standard file format called STL (short for STereoLithography). The STL format is a de facto standard that represents a 3D geometry by means of surfaces composed of a huge set of small triangles. In conventional software modeling systems, either the user is already working with triangles, or surfaces in 3D space are converted to triangles for STL export. This process often produces incomplete surfaces with cracks, holes and other issues. Unlike conventional models, the FRep models or functions we made to represent Escher's work result from a clear mathematical understanding of their surface that can be found by the evaluation and polygonalization of point sets.

The FRep objects, when evaluated and polygonized to capture the general details, generated a large amount of data in the form of STL files. Even with this large amount of data, some fine details of the surface have errors, such as the eyes (Fig. 5a) and the vertex of the spiral (Fig. 5b). This can be explained by the fact that we used a uniform, not an adaptive, polygonization, which could better recover small features of the surface. This led us to make several corrections in the file using special software for "stitching" or repairing STL files produced by conventional software. The problems detected in the STL files included open surfaces, bad edges, multiple shells, flipped triangles, boundary edges and holes. In the future, a better solution will be to control the digital fabrication process directly from the evaluation of the models, skipping the conversion to STL and driving the machines to their resolution limit for error-free details.

After the process of correcting the STL file, we decided to use two machines and processes for fabrication and compare the different technologies and results. We chose two popular commercial technologies: Selective Laser Sintering (SLS) and Fused Deposition Modelling (FDM) (see a brief tutorial at <www.additive3d.com/rp_int.htm>). The SLS system chosen was a Sinterstation HiQ from 3D Systems, using a polyamide powder material (nylon) called Duraform. For the FDM process, we used a Vantage I system from Stratasys running with a red Acrylonitrile Butadiene Styrene (ABS) plastic filament. Both machines have a controlled environment in the building chamber and proprietary software for processing the 3D models that uses STL files as input. The Sinterstation HiQ uses unsinterized powder as support for the part being constructed, while the Vantage I needs to deposit support in addition to the main model for the hanging structures of the part being constructed**.** The SLS process is much faster than FDM due to the fact that SLS is a laser scanning technique, while FDM uses a head with XYZ mechanical movement. Another advantage of SLS is that the support can easily be removed because it is just unsinterized powder, while in FDM the supports have to be removed mechanically and by special washing process. The fabricated objects can be seen in Figs. 6 and 7.

Conclusions

We have addressed several geometric modeling problems that arose from artworks by M.C. Escher and proposed original solutions to them. In addition to the geometric issues, there are many problems in representing complex forms for fabrication using the current standard file format. Another dilemma we need to solve is the huge amount of problematic data that is currently generated. To overcome this problem, we propose to use the direct fabrication of models and inter-operation of HyperFun modeling software, with AM machines as open hardware and software platforms [16].

Perhaps the most interesting aspect of this research, and the interplay between thought, space and dimension at the heart of Escher's artwork, is the relationships among the processes used in this work—the very direct relationship between mathematical functions and visualization and finally fabrication. Such physical fabrication from virtual objects can be considered a final closure of the long cycle from real physical constructions to the imaginary constructions of the possible to 2D illustrative reflections, to the virtual reality of abstract cyberworlds and finally back to physically materialized ones. Digital processes allow information, in this case in the form of the mathematical functions, to become active and tangible, as information can be directly represented physically. It no longer has to be interpreted by the human hand into the physical world with often-great effort (as seen in Escher's work alone), but instead has a direct, active and even relatively instantaneous link to and from the physical world.

References and Notes

Unedited references as provided by the authors.

1. Kaplan, C.S. (2008), Metamorphosis in Escher's art. Bridges 2008*: Mathematical Connections in Art, Music and Science*, pages 39–46, 2008.

2. Savransky, G., Dimerman D., Gotsman C. (1999), Modeling and rendering Escher-like impossible scenes, Computer Graphics Forum, vol. 18, 2:173–179.

3. Elber, G. (2009), Escher for Real, Technion, Israel Institute of Technology, Haifa, Israel, <www.cs.technion.ac.il/~gershon/EscherForReal/>.

4. Kaplan, C.S., Salesin, D.H. (2000), Escherization. SIGGRAPH 2000*, ACM,* 499–510.

5. Yen, J., Sequin, C. (2001), Escher sphere construction Kit, Symposium on Interactive 3D Graphics, ACM, 95—98.

6. Pasko, A., Adzhiev, V., Sourin, A., Savchenko, V. (1995), Function representation in geometric modeling: concepts, implementation and applications, The Visual Computer, vol. 11, 8:429–446.

7. Shape Modeling and Computer Graphics with Real Functions (2009), <www.hyperfun.org/F-rep.html>.

8. HyperFun Project (2009), <www.hyperfun.org>.

9. Rvachev, V.L. (1974), Methods of Logic Algebra in Mathematical Physics, Naukova Dumka, Kiev (in Russian).

10. Pasko, A., Adzhiev, V. (2004), Function-based shape modeling: Mathematical framework and specialized language, Automated Deduction in Geometry, Lecture Notes in Artificial Intelligence 2930, Ed. F. Winkler, Springer-Verlag, Berlin Heidelberg, 132–160.

11. Pasko, A., Savchenko, A., Savchenko, V. (1996), Implicit curved polygons, Technical Report 96-1-004, University of Aizu, Japan.

12. Pasko et al. [6].

13. Pasko, A., Savchenko, V., Sourin, A. (2001), Synthetic carving using implicit surface primitives, Computer-Aided Design, Elsevier, vol. 33, 5:379–388.

14. Pasko, G., Pasko, A. (2004), Trimming implicit surfaces, The Visual Computer, vol. 20, 7: 437–447.

15. Silva, J.V., Almeida, A.B., Raposo-do-Amaral, C.E., Ferreira, D., Hotta, L., Raposo-do-Amaral, C.A., Guidi, M.C., Buzzo, C.L. (2009), Three-dimensional virtual and physical technologies in the treatment of craniofacial anomalies, 11th International Congress on Cleft Lip and Palate and Related Craniofacial Anomalies—CLEFT, Medimond, International, Proceedings, 5–10.

16. Lixandrão Filho, A.L., Cheung, P.Y.C., Noritomi, P.Y., da Silva, J.V.L., Colangelo, N., Kang, H., Lipson, H., Butcher, J.T., Malone, E., and Inforçatti Neto, P. (2009), Construction and adaptation of an open source rapid prototyping machine for biomedical research purposes—a multinational collaborative development, Innovative Developments in Design and Manufacturing—Advanced Research in Virtual and Rapid Prototyping, Ed. P.J. Bartolo et al., CRC Press, Taylor & Francis.

Glossary

constructive solid geometry (CSG)—a solid modeling technique and an internal model representation based on logical (Boolean) combinations such as intersection, union and difference of predefined solid primitives (ball, cube, cone, cylinder, torus, etc.).

digital fabrication—a process of manufacturing a physical, tangible object under the control of a computer and on the basis of the given description of the object in the form of a digital data file.

function representation (FRep)—a generalization of traditional implicit surfaces, CSG, sweeping and other geometric models. Mathematically speaking, FRep defines a geometric object by the inequality $F(X) \geqq 0$, where F is a real function and X is a vector in n-dimensional Euclidean space. This inequality can be interpreted such that the function takes negative values outside the object it defines, positive values inside and zero value at the surface points of the object. This approach makes it possible to represent, by a single continuous function, such diverse objects as traditional skeleton-based implicit surfaces, convolution surfaces, constructive solids (using the so-called R-functions for set-theoretic operations), swept objects and volumetric objects. The advantage of this modeling approach is that the result of any supported operation can be treated as the input for a subsequent operation; thus very complex models can be created in this way from a single functional expression.

implicit surface—a surface in 3D space defined by the set of points possessing an equal value for a given continuous function of three variables (other terms: iso-surface, zero-level set, equipotential surface).

layered manufacturing—a process of manufacturing whereby material is added layer by layer and then adheres to the object being produced using glue, melting by heat, laser sintering and other methods (other terms: additive manufacturing, 3D printing).

metamorphosis—a process concerned with generating intermediate shapes between several "key-shapes," which traditionally requires the animator to establish correspondences between sets of points of the initial and final key-shapes of a metamorphosing object. This process is quite cumbersome and is incapable of dealing with key-shapes of differing topologies. In addition, using the traditional approach, it is difficult to obtain intermediate shapes by interpolating more than two key-shapes. In the FRep framework, metamorphosis is performed almost trivially using function interpolation. It is capable of handling key-shapes of differing topology and of generating intermediate shapes exhibiting genus change that may be composed of disjoint components.

STL file format—a standard format for representing surfaces for digital fabrication. The name stems from STereoLithography, one of the processes of layered manufacturing, where liquid resin is solidified by exposing it to the laser beam at the points, which have to be inside or at the surface of the manufactured object. The STL format represents the surface as a collection of disconnected triangles, which can easily introduce errors such as cracks and holes in the surfaces, intersecting triangles and incorrect normal vectors.

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